

Seminar: **Simplicial Topology**

# **Simplicial Approximation**

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# Preface

In the previous talk, we learned about simplicial complexes and maps. However, in the usual case we are starting with topological spaces and continuous maps between them. We already learned that it is possible to triangulate many topological spaces and by that obtain simplicial complexes. This raises the question if something similar is possible for continuous maps.

In this talk, we will introduce the notion of a *simplicial approximation* to a continuous map of the underlying topological spaces of some simplicial complexes. It is a simplicial map of the complexes which is “close” to the continuous map in some sense. We will see that it is not always possible to find such a simplicial map. However, we will show that it is possible to find subdivisions of the complexes which allow such an approximation. In order to achieve this, we will introduce *barycentric subdivision*, which will be our main tool.

For this, we will mainly follow [Mun84, §§14–16].

# Notation and Conventions

We will use the following notation and conventions:

- The non-negative integers are denoted by  $\mathbb{N}$  and the real numbers are denoted by  $\mathbb{R}$ .
- If  $A$  and  $B$  are sets, then  $A \subset B$  means that  $A$  is a subset of  $B$  and equality is permitted.
- If  $\sigma$  is a simplex, we denote the interior of  $\sigma$  by  $\text{Int } \sigma$  and the boundary of  $\sigma$  by  $\text{Bd } \sigma$ .
- We will often use the term “complex” instead of “geometric simplicial complex”.
- If  $J$  is a set, let  $\mathbb{R}^{(J)} = \bigoplus_J \mathbb{R}$ .
- In general, complexes lie in  $\mathbb{R}^{(J)}$  for some set  $J$ . If we work with *finite* complexes, we will mention it explicitly.
- If  $K$  is a complex, let  $|K|$  be its underlying topological space, equipped with the *weak topology*.
- If  $K$  is a complex, let  $K^{(p)}$  denote its  $p$ -skeleton.

# 1 Simplicial Approximation

In this chapter, we will define the notion of a simplicial approximation to a continuous map of the underlying spaces of two complexes and see some examples.

**Definition 1.1** (Star). Let  $K$  be a complex and let  $v$  be a vertex of  $K$ . The **star** of  $v$ , denoted by  $\text{St}(v, K)$  is defined to be the union of the interiors of those simplices of  $K$  that have  $v$  as a vertex. If it is clear which complex we are referring to, we sometimes also write  $\text{St } v$  instead of  $\text{St}(v, K)$ . The closure of this set is called the **closed star** of  $v$  and is denoted by  $\overline{\text{St}}(v, K)$  or  $\overline{\text{St}} v$ .

**Example 1.2.** Consider the vertex  $v$  of the complex pictured in Figure 1.1. By definition,  $\text{St } v$  is exactly the gray part of the complex.

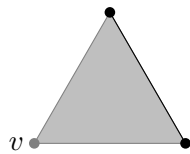


Figure 1.1: Star of a vertex of a 2-simplex

**Definition 1.3** (Star condition). Let  $K$  and  $L$  be complexes and let  $h: |K| \rightarrow |L|$  be a continuous map. We say  $h$  satisfies the **star condition** with respect to  $K$  and  $L$  if for each vertex  $v$  of  $K$ , there is a vertex  $w$  of  $L$  such that

$$h(\text{St } v) \subset \text{St } w.$$

**Lemma 1.4.** Let  $K$  and  $L$  be complexes and let  $h: |K| \rightarrow |L|$  satisfy the star condition with respect to  $K$  and  $L$ . Choose a map  $f: K^{(0)} \rightarrow L^{(0)}$  such that for each vertex  $v$  of  $K$ ,

$$h(\text{St } v) \subset \text{St } f(v).$$

- a) For  $\sigma \in K$  choose  $x \in \text{Int } \sigma$  and  $\tau \in L$  with  $h(x) \in \text{Int } \tau$ . Then  $f$  maps each vertex of  $\sigma$  to a vertex of  $\tau$ .
- b) The map  $f$  is a simplicial map  $K \rightarrow L$ .

## 1 Simplicial Approximation

*Proof.* a) Let  $\sigma = v_0 \dots v_p$ , then  $x \in \text{St } v_i$  for each  $i$  in  $\{0, \dots, p\}$ . Thus we have

$$h(x) \in h(\text{St } v_i) \subset \text{St } f(v_i).$$

Therefore  $h(x)$  has a positive barycentric coordinate with respect to each of the vertices  $f(v_i)$  for all  $i \in \{0, \dots, p\}$ . Hence these vertices form a subset of the vertex set of  $\tau$ .

b) By a),  $f$  carries the vertices of any simplex of  $K$  to the vertices of some simplex of  $L$  and thus is a simplicial map  $f: K \rightarrow L$ . □

**Definition 1.5** (Simplicial approximation). Let  $K$  and  $L$  be complexes and let  $h: |K| \rightarrow |L|$  be a continuous map. A simplicial map  $f: K \rightarrow L$  that satisfies

$$h(\text{St } v) \subset \text{St } f(v)$$

for each vertex  $v$  of  $K$  is called a **simplicial approximation** to  $h$ .

**Lemma 1.6.** *Let  $K$  and  $L$  be complexes and let  $f: K \rightarrow L$  be a simplicial approximation to the continuous map  $h: |K| \rightarrow |L|$ . Given  $x \in |K|$ , there is a simplex  $\tau$  of  $L$  such that  $h(x) \in \text{Int } \tau$  and  $f(x) \in \tau$ .*

*Proof.* This is an immediate consequence of a) of Lemma 1.4. □

**Theorem 1.7.** *Let  $K$ ,  $L$ , and  $M$  be complexes and let  $h: |K| \rightarrow |L|$  and  $k: |L| \rightarrow |M|$  be continuous maps. Let  $f: K \rightarrow L$  and  $g: L \rightarrow M$  be simplicial approximations to those maps, respectively. Then  $g \circ f$  is a simplicial approximation to  $k \circ h$ .*

*Proof.* Being the composition of two simplicial maps,  $g \circ f$  is a simplicial map. For each vertex  $v$  of  $K$  we have

$$h(\text{St } v) \subset \text{St } f(v),$$

because  $f$  is a simplicial approximation to  $h$ . Therefore

$$k(h(\text{St } v)) \subset k(\text{St } f(v)) \subset \text{St}(g(f(v))),$$

because  $g$  is a simplicial approximation to  $k$ . □

**Example 1.8.** Let  $P$  and  $Q$  be the complexes pictured in Figure 1.2, whose underlying topological spaces are homeomorphic to the circle and to the annulus, respectively. Let  $P'$  be the complex obtained from  $P$  by inserting extra vertices as pictured. Let  $h$  be the indicated continuous map, where we denote  $h(a)$  by  $A$  and similarly for the other vertices. One easily checks that  $h$  does not satisfy the star condition with respect to  $P$  and  $Q$ , but it does satisfy the star condition with respect to  $P'$  and  $Q$ . Thus it has a simplicial approximation  $f: P' \rightarrow Q$ . One such map is pictured. We denote  $f(a)$  by  $A'$  and similarly for the other vertices.

# 1 Simplicial Approximation

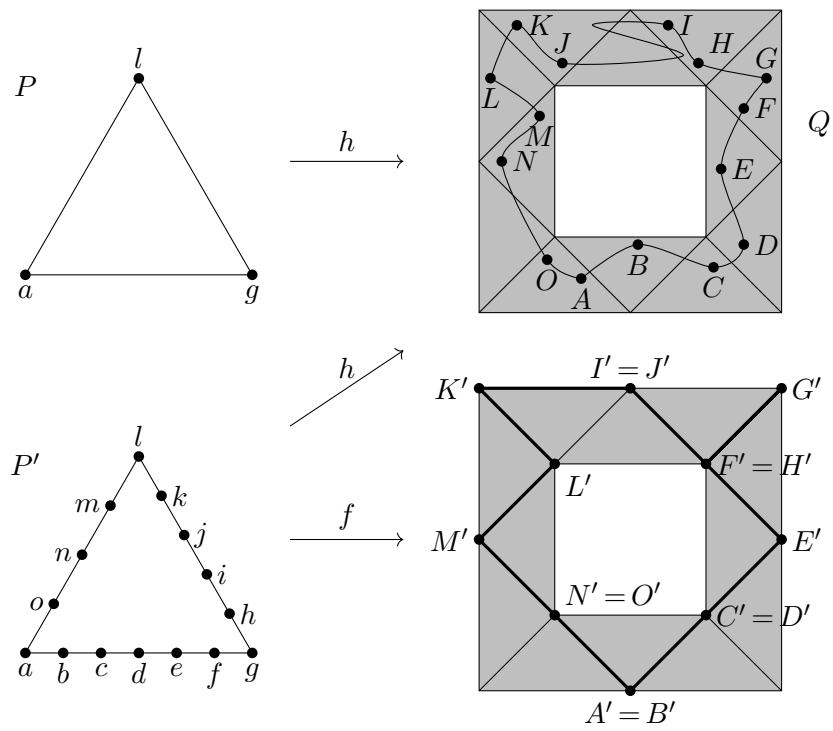


Figure 1.2: Example of a simplicial approximation

## 2 Barycentric Subdivision

In Example 1.8, we saw that if  $h: |K| \rightarrow |L|$  is a continuous map, there need not be a simplicial approximation  $K \rightarrow L$  to it. In the example, we were however able to get a simplicial approximation  $f: K' \rightarrow L$  to  $h$  by “subdividing”  $K$  into a simplicial complex  $K'$  with the same underlying topological space. In this chapter we will introduce a specific construction, which will allow us to subdivide finite complexes into simplices that are as small as desired. This construction is called *barycentric subdivision* and it will prove to be very helpful in the next chapter.

**Definition 2.1** (Subdivision). Let  $K$  be a complex. A complex  $K'$  is called a **subdivision** of  $K$  if:

- a) Each simplex of  $K'$  is contained in a simplex of  $K$ .
- b) Each simplex of  $K$  equals the union of *finitely* many simplices of  $K'$ .

**Lemma 2.2.** *Let  $K$  be a complex.*

- a) *If  $K'$  is a subdivision of  $K$ , then  $|K|$  and  $|K'|$  are equal as topological spaces.*
- b) *If  $K'$  is a subdivision of  $K$  and  $K''$  is a subdivision of  $K'$ , then  $K''$  is a subdivision of  $K$ .*

*Proof.* a) Obviously the union of the simplices of  $K'$  coincides with the union of the simplices of  $K$ . It follows that  $|K|$  and  $|K'|$  are equal as sets.

Let  $A$  be closed in  $K$  and let  $\tau \in K'$ . Then  $\tau \subset \sigma$  for some  $\sigma \in K$  and thus

$$A \cap \tau = (A \cap \sigma) \cap \tau.$$

But  $(A \cap \sigma)$  is closed in  $\sigma$ , by definition, so  $A \cap \tau$  is closed in  $\tau$ , because the topology on  $\tau$  is the subspace topology of  $\sigma$ . Therefore  $A$  is closed in  $K'$ .

Conversely, let  $A$  be closed in  $K'$  and let  $\sigma \in K$ . Then  $\sigma = \tau_1 \cup \dots \cup \tau_n$  for some  $\tau_1, \dots, \tau_n \in K'$  and thus

$$A \cap \sigma = A \cap (\tau_1 \cup \dots \cup \tau_n) = (A \cap \tau_1) \cup \dots \cup (A \cap \tau_n). \quad (2.1)$$

By definition, the set  $A \cap \tau_i$  is closed in  $\tau_i$ . Let  $m_i$  be the dimension of  $\tau_i$ , then  $A \cap \tau_i$  is closed in  $\mathbb{R}^{m_i+1}$ , because  $\tau_i$  is closed in  $\mathbb{R}^{m_i+1}$ . If  $m$  is the dimension of  $\sigma$ , then  $m \geq m_i$  and therefore  $A \cap \tau_i$  is closed in  $\mathbb{R}^{m+1}$ . But  $\sigma$  carries the subspace topology of  $\mathbb{R}^{m+1}$ , thus  $A \cap \tau_i$  is closed in  $\sigma$ . Hence, by (2.1),  $A \cap \sigma$  is the finite union of closed sets and thus is closed.



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b) This is an immediate consequence of the conditions in Definition 2.1. □

**Lemma 2.3.** *Let  $K$  be a complex and let  $K'$  be a subdivision of  $K$ . Then for each vertex  $w$  of  $K'$ , there is a vertex  $v$  of  $K$  such that*

$$\text{St}(w, K') \subset \text{St}(v, K).$$

*Indeed, if  $\sigma$  is the simplex of  $K$  such that  $w \in \text{Int } \sigma$ , then this inclusion holds if and only if  $v$  is a vertex of  $\sigma$ .*

*Proof.* If this inclusion holds,  $w$  lies in some open simplex which has  $v$  as a vertex, because  $w \in \text{St}(w, K')$ .

Conversely, suppose  $w \in \text{Int } \sigma$  and  $v$  is a vertex of  $\sigma$ . It suffices to show that

$$|K| \setminus \text{St}(v, K) \subset |K| \setminus \text{St}(w, K').$$

The set on the left side of the inclusion is the union of those simplices of  $K$  that do not have  $v$  as a vertex. Thus it is also the union of some simplices of  $K'$ . No such simplex can have  $w$  as a vertex, because  $w \in \text{Int } \sigma \subset \text{St}(v, K)$ . Hence any such simplex lies in  $|K| \setminus \text{St}(w, K')$ . □

**Lemma 2.4.** *If  $K$  is a complex, then the intersection of any collection of subcomplexes of  $K$  is a subcomplex of  $K$ . Conversely, if  $(K_\alpha)_{\alpha \in I}$  is a collection of complexes and if the intersection  $|K_\alpha| \cap |K_\beta|$  is the polytope of a subcomplex of both  $K_\alpha$  and  $K_\beta$  for every pair  $\alpha, \beta$ , then the union  $\bigcup_{\alpha \in I} K_\alpha$  is a complex.*

*Proof.* This follows immediately from the definition of subcomplexes. □

**Lemma/Definition 2.5 (Cone).** *Let  $K$  be a complex and let  $w$  be a point of  $\mathbb{R}^{(J)}$  such that each ray emanating from  $w$  intersects  $|K|$  in at most one point. We define the **cone on  $K$  with vertex  $w$**  to be the complex consisting of all simplices of the form  $wa_0 \dots a_p$  where  $a_0 \dots a_p$  is a simplex of  $K$ , along with the faces of all such simplices. We denote this complex by  $w * K$ . The complex  $K$  is called the **base** of the cone and it is actually a subcomplex of  $w * K$ .*

*Proof.* First we show that the set  $w, a_0, \dots, a_p$  is geometrically independent for all simplices  $a_0 \dots a_p$  of  $K$ . If  $w$  was in the plane  $P$  determined by  $a_0, \dots, a_p$ , we could consider the line segment joining  $w$  and an interior point  $x$  of  $\sigma = a_0 \dots a_p$ . Since  $\text{Int } \sigma$  is open in  $P$ , it would contain an interval of points of the line segment contradicting the fact that, by hypothesis, the ray from  $w$  through  $x$  intersects  $|K|$  in only one point.

Now we show that  $w * K$  is a complex. There are three types of simplices in  $w * K$ :

- Simplices  $a_0 \dots a_p$  of  $K$ .
- Simplices of the form  $wa_0 \dots a_p$  for a simplex  $a_0 \dots a_p$  of  $K$ .

## 2 Barycentric Subdivision

- The 0-simplex  $w$ .

Because  $K$  is a complex, any two simplices of the first type have disjoint interiors. The open simplex  $\text{Int}(wa_0 \dots a_p)$  is the union of all open line segments joining  $w$  to points of  $\text{Int}(a_0 \dots a_p)$ . No two such open simplices can intersect, because no ray from  $w$  contains more than one point of  $|K|$ . For the same reason, simplices of the first and second types have disjoint interiors. Thus any two simplices of  $w * K$  have disjoint interiors, which makes  $w * K$  a complex.

We have already seen above that all simplices of  $K$  are also simplices of  $w * K$ , so  $K$  is a subcomplex of  $w * K$ .  $\square$

**Lemma/Definition 2.6** (Starring). *Let  $K$  be a complex and  $L_p$  a subdivision of the  $p$ -skeleton of  $K$ . If  $\sigma$  is a  $(p+1)$ -simplex of  $K$ , the space  $\text{Bd } \sigma$  is the polytope of a subcomplex of the  $p$ -skeleton of  $K$ . Thus it is also the polytope of a subcomplex  $L_\sigma$  of  $L_p$ . For  $w_\sigma \in \text{Int } \sigma$  the cone  $w_\sigma * L_\sigma$  is a complex whose underlying space is  $\sigma$ . Now for each  $\sigma \in K^{(p+1)}$ , let  $w_\sigma \in \text{Int } \sigma$ . Then we define  $L_{p+1}$  to be the union of  $L_p$  and the complexes  $w_\sigma * L_\sigma$  for all  $(p+1)$ -simplices  $\sigma$  of  $K$ . We call  $L_{p+1}$  the **subdivision of  $K^{(p+1)}$  obtained by starring  $L_p$  from the points  $w_\sigma$** .*

*Proof that  $L_{p+1}$  is a complex.* We first note that

$$|w_\sigma * L_\sigma| \cap |L_p| = \text{Bd } \sigma.$$

Thus  $|w_\sigma * L_\sigma| \cap |L_p|$  is the polytope of  $L_\sigma$  which is a subcomplex of both  $w_\sigma * L_\sigma$  and  $L_p$ . If  $\tau$  is another  $(p+1)$ -simplex of  $K$ , we have

$$|w_\tau * L_\tau| \cap |w_\sigma * L_\sigma| = \tau \cap \sigma.$$

Because  $\sigma \cap \tau$  is a face of both  $\sigma$  and  $\tau$ , the space  $|w_\sigma * L_\sigma| \cap |w_\tau * L_\tau|$  is the polytope of a subcomplex of both  $w_\sigma * L_\sigma$  and  $w_\tau * L_\tau$ . So  $L_{p+1}$  is a complex by Lemma 2.4.  $\square$

In the above definition, the complex  $L_{p+1}$  depends on the choice of the points  $w_\sigma$ . The usual point used in this case is the following:

**Definition 2.7** (Barycenter). For  $\sigma = v_0 \dots v_p$ , the **barycenter** of  $\sigma$  is defined as

$$\hat{\sigma} = \sum_{i=0}^p \frac{1}{p+1} v_i.$$

It is the point of  $\text{Int } \sigma$  whose barycentric coordinates with respect to  $\sigma$  are all equal.

**Definition 2.8** (Barycentric subdivision). Let  $K$  be a complex. We define a sequence of subdivisions of the skeletons of  $K$  as follows: Let  $L_0 = K^{(0)}$ . For a subdivision  $L_p$  of the  $p$ -skeleton of  $K$  let  $L_{p+1}$  be the subdivision of  $K^{(p+1)}$  obtained by starring  $L_p$  from the barycenters of the  $(p+1)$ -simplices of  $K$ . The union of these complexes is called the

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**first barycentric subdivision** of  $K$  and it is denoted by  $\text{sd } K$ . By Lemma 2.4,  $\text{sd } K$  is a complex.

The  $n$ -th **barycentric subdivision** of  $K$  is defined to be  $\text{sd}^n K := \underbrace{\text{sd} \dots \text{sd}}_{n\text{-times}} K$ .

**Example 2.9.** Consider the complex  $K$  as pictured in Figure 2.1. Constructing its first barycentric subdivision results in the complex  $\text{sd } K$ .

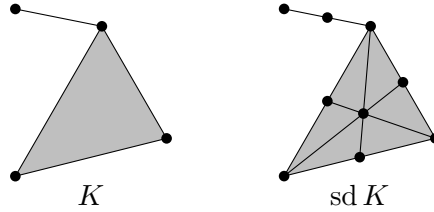


Figure 2.1: Barycentric subdivision

Sometimes we need a more explicit description of the simplices of the first barycentric subdivision. The following lemma contains such a description. If  $\sigma_2$  is a proper face of  $\sigma_1$ , let us use the notation  $\sigma_2 \prec \sigma_1$ .

**Lemma 2.10.** *Let  $K$  be a complex. Then  $\text{sd } K$  equals the collection of all simplices of the form*

$$\hat{\sigma}_1 \hat{\sigma}_2 \dots \hat{\sigma}_n,$$

with  $\sigma_n \prec \dots \prec \sigma_2 \prec \sigma_1$ .

*Proof.* We prove this by induction over the skeleta. It is obvious that simplices of  $\text{sd } K$  lying in the subdivision of  $K^{(0)}$  are of this form, because they are vertices of  $K$  and  $\hat{v} = v$  for a vertex  $v$ .

Now suppose each simplex of  $\text{sd } K$  lying in  $|K^{(p)}|$  is of this form. Let  $\tau$  be a simplex of  $\text{sd } K$  lying in  $|K^{(p+1)}|$  but not in  $|K^{(p)}|$ . Then there is a  $(p+1)$ -simplex  $\sigma$  of  $K$  such that  $\tau$  is a simplex of  $\hat{\sigma} * L_\sigma$ , where  $L_\sigma$  is the first barycentric subdivision of the complex consisting of the proper faces of  $\sigma$ . By induction hypothesis, each simplex of  $L_\sigma$  is of the form  $\hat{\sigma}_1 \hat{\sigma}_2 \dots \hat{\sigma}_n$  with  $\sigma_n \prec \dots \prec \sigma_2 \prec \sigma_1$  and  $\sigma_1$  is a proper face of  $\sigma$ . But then  $\tau$  is of the form

$$\hat{\sigma} \hat{\sigma}_1 \hat{\sigma}_2 \dots \hat{\sigma}_n$$

as desired. □

With this result, one can easily define barycentric subdivision for *abstract* simplicial complexes.

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**Theorem 2.11.** *Let  $K$  be a finite complex and let  $|K|$  be equipped with a metric that generates its topology. Given  $\varepsilon > 0$ , there is an  $N \in \mathbb{N}$  such that each simplex of  $\text{sd}^N K$  has diameter less than  $\varepsilon$ .*

*Proof.* Because  $K$  is finite,  $|K|$  is a subspace of the euclidean space  $\mathbb{R}^{(J)}$ , in which it lies. If  $d_1$  and  $d_2$  are two metrics for  $|K|$  generating its topology, they are equivalent, hence the identity maps  $(|K|, d_1) \rightarrow (|K|, d_2)$  and  $(|K|, d_2) \rightarrow (|K|, d_1)$  are continuous. Therefore, because  $|K|$  is compact, they are uniformly continuous, by the *uniform continuity theorem* [Mun00, Theorem 27.6]. Thus  $d_1$  and  $d_2$  are uniformly equivalent, and hence, given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that any subset of  $|K|$  with  $d_1$ -diameter less than  $\delta$  has  $d_2$ -diameter less than  $\varepsilon$  and vice versa. Therefore, we may as well use the metric of  $\mathbb{R}^{(J)}$ , which is

$$|x - y| = \max |x_j - y_j|.$$

This also shows that a metric on  $|K|$ , as required, always exists.

*Step 1.* We show that the diameter of a simplex  $\sigma = v_0 \dots v_p$  is equal to the number  $l = \max(|v_i - v_j|)$  which is the maximum distance between the vertices of  $\sigma$ . Because  $v_i, v_j \in \sigma$ , we know  $\text{diam } \sigma \geq l$ , so it remains to show the reverse inequality.

We first show that  $|x - v_i| \leq l$  for every  $x \in \sigma$  and every  $i \in \{0, \dots, p\}$ . Let  $i \in \{0, \dots, p\}$  and consider the closed ball  $\overline{B_l(v_i)}$  with radius  $l$  and centre  $v_i$ . It is convex and thus, because it contains all vertices of  $\sigma$ , it contains  $\sigma$ . So  $|x - v_i| \leq l$  for all  $x \in \sigma$ .

Now let us show that  $|x - z| \leq l$  for all  $x, z \in \sigma$ . Given  $x \in \sigma$ , consider the closed ball  $\overline{B_l(x)}$ . By the preceding result, this set contains all vertices of  $\sigma$  and thus, because it is convex, it contains  $\sigma$ . Hence  $|x - z| \leq l$  for  $x, z \in \sigma$  and we get  $\text{diam } \sigma = l$  as desired.

*Step 2.* We now show that, if  $\sigma$  has dimension  $p$ , then for all  $z \in \sigma$ , we have

$$|\hat{\sigma} - z| \leq \frac{p}{p+1} \text{diam } \sigma.$$

To prove this, we compute

$$\begin{aligned} |v_j - \hat{\sigma}| &= \left| v_j - \sum_{i=0}^p \frac{1}{p+1} v_i \right| \\ &= \left| \sum_{i=0}^p \frac{1}{p+1} (v_j - v_i) \right| \\ &= \left| \sum_{i=0, i \neq j}^p \frac{1}{p+1} (v_j - v_i) \right| \end{aligned}$$

## 2 Barycentric Subdivision

$$\begin{aligned}
&\leq \frac{1}{p+1} \sum_{i=0, i \neq j}^p |(v_j - v_i)| \\
&\leq \frac{p}{p+1} \max |v_j - v_i| \\
&\leq \frac{p}{p+1} \text{diam } \sigma.
\end{aligned}$$

This shows that the closed ball of radius  $\frac{p}{p+1} \text{diam } \sigma$  with centre  $\hat{\sigma}$  contains all vertices of  $\sigma$  and, because it is convex, it contains  $\sigma$ .

*Step 3.* We show that, if  $\sigma$  is a  $p$ -simplex and  $\tau$  is a simplex in the first barycentric subdivision of  $\sigma$ , then

$$\text{diam } \tau \leq \frac{p}{p+1} \text{diam } \sigma.$$

We prove this by induction. For  $p = 0$  the result is trivial, because 0-simplices are vertices and the first barycentric subdivision of a vertex is simply the vertex itself. Suppose now that the above equation is true in dimensions less than  $p$ . Let  $\sigma$  be a  $p$ -simplex and  $\tau$  be a simplex in the first barycentric subdivision of  $\sigma$ . By Lemma 2.10,  $\tau$  is of the form

$$\hat{\sigma}_1 \dots \hat{\sigma}_n,$$

where  $\sigma_1, \dots, \sigma_n$  are faces of  $\sigma$  with  $\sigma_n \prec \dots \prec \sigma_1$ . According to Step 1, we know

$$\text{diam } \tau = \max |\hat{\sigma}_i - \hat{\sigma}_j|,$$

so it suffices to show that for two faces  $s$  and  $s'$  of  $\sigma$  with  $s' \prec s$ , we have

$$|\hat{s} - \hat{s}'| \leq \frac{p}{p+1} \text{diam } \sigma.$$

For  $s = \sigma$  this follows from Step 2. If  $s$  is a proper face of  $\sigma$  of dimension  $q$ , then

$$|\hat{s} - \hat{s}'| \leq \frac{q}{q+1} \text{diam } s \leq \frac{p}{p+1} \text{diam } \sigma.$$

The first inequality follows by the induction hypothesis while the second follows from the fact that  $x \mapsto \frac{x}{x+1}$  is a monotonically increasing function for  $x \geq 0$ .

*Step 4.* Let  $K$  be of dimension  $n$  and let  $d$  be the maximum diameter of any simplex of  $K$ . Because the maximum dimension of any simplex in  $K$  is  $n$  and  $n \mapsto \frac{n}{n+1}$  is monotonically increasing for  $n \in \mathbb{N}$ , we know that the maximum diameter of a simplex in the  $N$ -th barycentric subdivision of  $K$  is at most  $(\frac{n}{n+1})^N d$ . But this number is less than  $\varepsilon$  if  $N$  is large enough.  $\square$

### 3 The Simplicial Approximation Theorem

We now show that if  $K$  and  $L$  are two complexes and  $h: |K| \rightarrow |L|$  is a continuous map, then there exists a subdivision  $K'$  of  $K$  such that there is a simplicial approximation  $K' \rightarrow L$  to  $h$ . In the case that  $K$  is finite, this follows easily by using barycentric subdivision. However, in the general case, a more sophisticated technique of subdivision, called the generalized barycentric subdivision, is required.

**Theorem 3.1** (The finite simplicial approximation theorem). *Let  $K$  and  $L$  be complexes and let  $K$  be finite. Given a continuous map  $h: |K| \rightarrow |L|$ , there is an  $N \in \mathbb{N}$  such that  $h$  has a simplicial approximation  $\text{sd}^N K \rightarrow L$ .*

*Proof.* The set  $\mathcal{A} = \{h^{-1}(\text{St}(w, L)) \mid w \in L^{(0)}\}$  is an open covering of  $|K|$ . Because  $|K|$  is a compact metric space, by the *Lebesgue number lemma* [Mun00, Lemma 27.5], there is a number  $\lambda > 0$  such that every subset of  $K$  with diameter less than  $\lambda$  is contained in one of the elements of  $\mathcal{A}$ . The number  $\lambda$  is called a **Lebesgue number** for  $\mathcal{A}$ .

Now choose  $N$  such that each simplex of  $\text{sd}^N K$  has diameter less than  $\frac{\lambda}{2}$ . Then the diameter of each star of a vertex of  $\text{sd}^N K$  is less than  $\lambda$  and thus lies in  $h^{-1}(\text{St } w)$  for some vertex  $w$  of  $L$ . But then  $h: |K| \rightarrow |L|$  satisfies the star condition with respect to  $\text{sd}^N K$  and  $L$  and, by Lemma 1.4, the desired approximation exists.  $\square$

**Theorem 3.2** (The general simplicial approximation theorem). *Let  $K$  and  $L$  be complexes and let  $h: K \rightarrow L$  be a continuous map. There is a subdivision  $K'$  of  $K$  such that  $h$  has a simplicial approximation  $K' \rightarrow L$ .*

In order to prove this, we need some more techniques. As a first step, we will show how to subdivide a complex  $K$  in a way that leaves a given subcomplex  $K_0$  of  $K$  unchanged.

**Definition 3.3** (Barycentric subdivision, holding a subcomplex fixed). Let  $K$  be a complex and let  $K_0$  be a subcomplex of  $K$ . We define a sequence of subdivisions of the skeletons of  $K$  as follows: Let  $J_0 = K^{(0)}$ . In general, suppose  $J_p$  is a subdivision of the  $p$ -skeleton of  $K$  and each simplex of  $K_0$  of dimension at most  $p$  is also a simplex of  $J_p$ . If  $\sigma$  is a  $(p+1)$ -simplex of  $K$ , let  $J_\sigma$  be the subcomplex of  $J_p$  whose polytope is  $\text{Bd } \sigma$ . Then let  $J_{p+1}$  be the union of  $J_p$ , all  $(p+1)$ -simplices of  $K_0$ , and the complexes  $\hat{\sigma} * J_\sigma$  for all  $(p+1)$ -simplices of  $K$  not in  $K_0$ . The union of the complexes  $J_p$  is a subdivision

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of  $K$  which is called the **first barycentric subdivision of  $K$ , holding  $K_0$  fixed** and is denoted by  $\text{sd}(K, K_0)$ .

As with barycentric subdivision, we can repeat this process to get the  **$n$ -th barycentric subdivision of  $K$ , holding  $K_0$  fixed**, which we denote by  $\text{sd}^n(K, K_0)$ .

**Example 3.4.** Consider the complex  $K$  and its subcomplex  $K_0$  as pictured in Figure 3.1. Constructing the first barycentric subdivision of  $K$  holding  $K_0$  fixed results in the complex  $\text{sd}(K, K_0)$ .

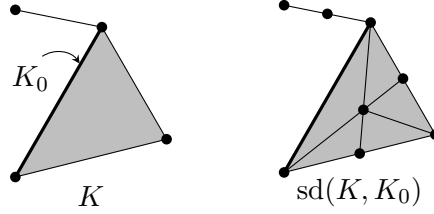


Figure 3.1: Barycentric subdivision, holding a subcomplex fixed

The reason that barycentric subdivision does not suffice in the general case is the fact that the Lebesgue number argument in the proof of Theorem 3.1 requires  $|K|$  to be compact. For a general complex, the number  $\lambda$  which measures how finely a simplex must be subdivided may vary between the different simplices. We will now generalize our notion of barycentric subdivision to accommodate for this.

**Definition 3.5** (Generalized barycentric subdivision). Let  $K$  be a complex. Let  $N$  be a function assigning to each simplex of  $K$  with positive dimension, a number  $N(\sigma) \in \mathbb{N}$ . We construct a subdivision of  $K$  as follows: Let  $L_0 = K^{(0)}$ . In general, suppose  $L_p$  is a subdivision of the  $p$ -skeleton of  $K$ . For each  $(p+1)$ -simplex of  $K$ , let  $L_\sigma$  be the subcomplex of  $L_p$  whose polytope is  $\text{Bd } \sigma$ . Now let  $L_{p+1}$  be the union of  $L_p$  and the complexes

$$\text{sd}^{N(\sigma)}((\hat{\sigma} * L_\sigma), L_\sigma),$$

as  $\sigma$  ranges over all  $(p+1)$ -simplices of  $K$ . Then  $L_{p+1}$  is a subdivision of the  $(p+1)$ -skeleton of  $K$ . The union of the complexes  $L_p$  is a subdivision of  $K$  called the **generalized barycentric subdivision** of  $K$  corresponding to the function  $N(\sigma)$ .

The following lemma shows the difference between  $\text{sd } K$  and  $\text{sd}(K, K_0)$  for a complex  $K$  and a subcomplex  $K_0$  of  $K$ .

**Lemma 3.6.** *Let  $K$  be a complex and let  $K_0$  be a subcomplex of  $K$ .*

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a) If  $\tau$  is a simplex of  $\text{sd}(K, K_0)$ , then  $\tau$  is of the form

$$\tau = \hat{\sigma}_1 \dots \hat{\sigma}_q v_0 \dots v_p,$$

where  $s = v_0 \dots v_p$  is a simplex of  $K_0$  and  $\sigma_1, \dots, \sigma_q$  are simplices of  $K$  not in  $K_0$  with  $s \prec \sigma_q \prec \dots \prec \sigma_1$ .

b) Either  $v_0 \dots v_p$  or  $\hat{\sigma}_1 \dots \hat{\sigma}_q$  may be missing from this expression. The simplex  $\tau$  is disjoint from  $|K_0|$  if and only if  $v_0 \dots v_p$  is missing. In this case,  $\tau$  is a simplex of  $\text{sd } K$ .

*Proof.* a) Let  $J_p$  as in Definition 3.3 for all  $p \leq \dim K$ . The result is true for  $\tau \in J_0$ . In general, let  $\tau$  be a simplex of  $J_{p+1}$  that is not in  $J_p$ . Then either  $\tau$  belongs to  $K_0$ , in which case  $\tau$  is of the form  $v_0 \dots v_r$ , or  $\tau$  belongs to one of the cones  $\hat{\sigma} * J_\sigma$ . By the induction hypothesis, each simplex of  $J_\sigma$  is of the form  $\hat{\sigma}_1 \dots \hat{\sigma}_q v_0 \dots v_r$ , where  $\sigma_1 \prec \sigma$ . But then  $\tau = \hat{\sigma} \hat{\sigma}_1 \dots \hat{\sigma}_q v_0 \dots v_r$ , as desired.

b) Let  $\tau = \hat{\sigma}_1 \dots \hat{\sigma}_q v_0 \dots v_p$ . If  $v_0 \dots v_p$  is not missing from this expression, then  $\tau$  intersects  $|K_0|$  in  $v_0 \dots v_p$ , at least. Conversely, if the set  $\tau \cap |K_0|$  is non-empty, then it contains a face of  $\tau$  and thus a vertex of  $\tau$ . Since none of the points  $\hat{\sigma}_1, \dots, \hat{\sigma}_q$  is in  $|K_0|$ , the term  $v_0 \dots v_p$  cannot be missing.  $\square$

To prove Theorem 3.2 we need to show that given a continuous map  $h: |K| \rightarrow |L|$ , there is a subdivision  $K'$  of  $K$  such that  $h$  satisfies the star condition with respect to  $K'$  and  $L$ . This is equivalent to the following statement: If  $\mathcal{A}$  is the open covering of  $|K|$  defined by

$$\mathcal{A} = \{h^{-1}(\text{St}(w, L)) \mid w \in L^{(0)}\},$$

then there is a subdivision  $K'$  of  $K$  such that the collection

$$\mathcal{B} = \{\text{St}(v, K') \mid v \in K'^{(0)}\}$$

refines  $\mathcal{A}$ , that is each element of  $\mathcal{B}$  is contained in some element of  $\mathcal{A}$ .

We will actually prove a slightly stronger statement. We will construct a subdivision  $K'$  of  $K$ , which is fine enough that the collection  $\{\overline{\text{St}}(v, K') \mid v \in K'^{(0)}\}$  of closed stars in  $K'$  refines  $\mathcal{A}$ .

The following lemma is a big step towards this goal, because it will allow us to carry out the induction step of the proof.

**Lemma 3.7.** *Let  $K = p * B$  be a cone over a finite complex  $B$ . Let  $\mathcal{A}$  be an open covering of  $|K|$ . Suppose for each vertex  $v$  of  $B$  there is an element  $A_v$  of  $\mathcal{A}$  such that*

$$\overline{\text{St}}(v, B) \subset A_v.$$

*Then there is an  $N \in \mathbb{N}$  such that the collection of closed stars of  $\text{sd}^N(K, B)$  refines  $\mathcal{A}$ , and furthermore such that for each vertex  $v$  of  $B$*

$$\overline{\text{St}}(v, \text{sd}^N(K, B)) \subset A_v.$$



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*Proof.* Since  $B$  is a finite complex,  $|B|$  lies in  $\mathbb{R}^m \times \{0\}$  for some  $m$ . Without loss of generality we assume  $p = (0, \dots, 0, 1) \in \mathbb{R}^m \times \mathbb{R}$ . Let  $n = \dim K$ .

*Step 1.* In general, the maximum diameter of the simplices of  $\text{sd}^N(K, B)$  does not go to zero as  $N$  increases. For if  $\sigma \in K$  has a face in  $B$  with positive dimension, that face never gets subdivided. However, as  $N$  increases, the simplices that intersect the plane  $\mathbb{R}^m \times \{0\}$  lie closer and closer to this plane. More generally, we show: If  $K'$  is any subdivision of  $K$  that keeps  $B$  fixed and if the simplices of  $K'$  that intersect  $\mathbb{R}^m \times \{0\}$  lie in the strip  $\mathbb{R}^m \times [0, \varepsilon]$  for  $\varepsilon > 0$ , then any simplex  $\tau$  of  $\text{sd}(K', B)$  that intersects  $\mathbb{R}^m \times \{0\}$  lies in the strip  $\mathbb{R}^m \times [0, \frac{n}{n+1}\varepsilon]$ .

Now  $\tau$  is of the form  $\hat{\sigma}_1 \dots \hat{\sigma}_q v_0 \dots v_p$  as in Lemma 3.6. If  $\tau$  intersects  $\mathbb{R}^m \times \{0\}$ , but does not lie in it, neither  $\hat{\sigma}_0 \dots \hat{\sigma}_q$  nor  $v_0 \dots v_p$  are missing from this expression. Additionally, each vertex  $v_i$  lies in  $\mathbb{R}^m \times \{0\}$ .

Consider the vertex  $\hat{\sigma}_j$  of  $\tau$ . The simplex  $\sigma_j$  of  $K'$  intersects  $\mathbb{R}^m \times \{0\}$ , because  $\sigma_j$  has  $v_0, \dots, v_p$  as a face. Therefore  $\sigma_j \subset \mathbb{R}^m \times [0, \varepsilon]$ . Let  $w_0, \dots, w_k$  be the vertices of  $\sigma$  and let  $\pi: \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$  be the projection onto the last coordinate. Then  $\pi(w_i) \leq \varepsilon$  for all  $i \in \{0, \dots, k\}$  and  $\pi(w_i) = 0$  for at least one  $i \in \{0, \dots, k\}$ . Therefore

$$\pi(\hat{\sigma}_j) = \sum_{i=0}^k \binom{k}{i} \pi(w_i) \leq \binom{k}{k+1} \varepsilon.$$

Thus each vertex of  $\tau$  lies in  $\mathbb{R}^m \times [0, \frac{n}{n+1}\varepsilon]$  and, because this set is convex, so does all of  $\tau$ .

*Step 2.* Let  $K_N$  denote the complex  $\text{sd}^N(K, B)$ . We show that there is an  $N_0 \in \mathbb{N}$  such that if  $N \geq N_0$ , then for each  $v \in B^{(0)}$ ,

$$\overline{\text{St}}(v, K_N) \subset A_v. \tag{3.1}$$

To prove this, consider the continuous map

$$\rho: |B| \times I \rightarrow |K|, (x, t) \mapsto (1-t)x + tp.$$

Because  $K$  is the cone on  $B$  with vertex  $p$ , it carries  $\overline{\text{St}}(v, B) \times I$  onto  $\overline{\text{St}}(v, K)$ . Furthermore, since

$$(\pi \circ \rho)(x, t) = (1-t)\pi(x) + t\pi(x) = (1-t) \cdot 0 + t \cdot 1 = \pi(x, t),$$

it preserves the last coordinate. The set  $\overline{\text{St}}(v, B)$  is compact, so, by the *tube lemma* [Mun00, Lemma 26.8], there is a  $\delta > 0$  such that

$$\overline{\text{St}}(v, B) \times [0, \delta] \subset \rho^{-1}(A_v).$$

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Therefore the set

$$\overline{\text{St}}(v, K) \cap (\mathbb{R}^m \times [0, \delta]) = \rho(\overline{\text{St}}(v, B) \times [0, \delta])$$

lies in  $A_v$ .

By applying Step 1, we choose  $N_0 \in \mathbb{N}$  such that for  $N \geq N_0$ , each simplex of  $K_N$  that intersects  $\mathbb{R}^m \times \{0\}$  lies in  $\mathbb{R}^m \times [0, \delta]$ . If  $v \in B^{(0)}$ , the set  $\overline{\text{St}}(v, K_N)$  lies in  $\mathbb{R}^m \times [0, \delta]$ , but this set also lies in  $\overline{\text{St}}(v, K)$ , so it is a subset of  $A_v$ , as desired.

*Step 3.* Now let  $N_0$  be as in Step 2. Consider the complex  $K_{N_0+1}$ . Let  $P$  be the union of all simplices of  $K_{N_0+1}$  that intersect  $B$  and let  $Q$  be the union of all simplices of  $K_{N_0+1}$  that do not intersect  $B$ . We claim the following: If  $N \geq N_0 + 1$ , then for each  $w \in K_N^{(0)}$  with  $w \in P$  and  $w \notin |B|$ , there is an  $A \in \mathcal{A}$  such that

$$\overline{\text{St}}(w, K_N) \subset A. \quad (3.2)$$

We first prove this in the case  $N = N_0 + 1$ . The space  $P$  is the polytope of a subcomplex of  $K_{N_0+1}$ , by definition. If  $w \in K_N^{(0)}$  with  $w \in P$  and  $w \notin |B|$ , then, by Lemma 3.6,  $w = \hat{\sigma}$  for some simplex  $\sigma$  of  $K_{N_0}$  that intersects  $B$  but does not lie in  $B$ . Let  $v$  be a vertex of  $\sigma$  lying in  $B$ . Because  $w \in \text{Int } \sigma$ , we have

$$\text{St}(w, K_{N_0+1}) \subset \text{St}(v, K_{N_0}),$$

by Lemma 2.3. Then

$$\overline{\text{St}}(w, K_{N_0+1}) \subset \overline{\text{St}}(v, K_{N_0}) \subset A_v,$$

by (3.1).

Now we prove (3.2) in the case  $N > N_0 + 1$ . If  $w' \in K_N^{(0)}$  with  $w' \in P$ , then  $w' \in \text{St}(w, K_{N_0+1})$  for some vertex  $w$  of  $K_{N_0+1}$  lying in  $P$ . By Lemma 2.3, we have  $\text{St}(w', K_N) \subset \text{St}(w, K_{N_0+1})$ . Therefore

$$\overline{\text{St}}(w', K_N) \subset \overline{\text{St}}(w, K_{N_0+1}) \subset A_v,$$

as desired.

*Step 4.* Let  $\lambda$  be a Lebesgue number for  $\mathcal{A}$ . Consider the space  $Q$ . It is the polytope of a subcomplex  $J$  of  $K_{N_0+1}$ . In forming the subdivision  $K_{N_0+2}$ , each simplex of  $J$  is subdivided barycentrically, by Lemma 3.6. Thus  $\text{sd } J$  is a subcomplex of  $K_{N_0+2}$ . By repeating this argument, we see that in general  $\text{sd}^M J$  is a subcomplex of  $K_{N_0+M}$ .

Now choose  $M$  large enough that the diameter of each simplex of  $\text{sd}^M J$  is less than  $\frac{\lambda}{2}$ . If  $N \geq N_0 + 1 + M$  and  $w \in K_N^{(0)}$  with  $w \notin P$ , we claim there is an  $A \in \mathcal{A}$  such that

$$\overline{\text{St}}(w, K_N) \subset A. \quad (3.3)$$

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This claim is true, for if  $w \notin P$ , each simplex of  $K_N$  having  $w$  as a vertex lies in  $Q$  and thus is a simplex of  $\text{sd}^M J$ . Therefore  $\overline{\text{St}}(w, K_n)$  has diameter less than  $\lambda$  and hence is contained in an element of  $\mathcal{A}$ .

The combination of (3.1), (3.2), and (3.3) proves the lemma.  $\square$

**Theorem 3.8.** *Let  $K$  be a complex and let  $\mathcal{A}$  be an open covering of  $|K|$ . There exists a generalized barycentric subdivision  $K'$  of  $K$  such that the collection*

$$\{\overline{\text{St}}(w, K') \mid w \in K'^{(0)}\}$$

*of closed stars refines  $\mathcal{A}$ .*

*Proof.* We proceed step-by-step. Initially, let  $L_0 = K^{(0)}$  and for each  $v \in K^{(0)}$  choose an  $A_v \in \mathcal{A}$  with  $v \in A_v$ .

In general, we assume a subdivision  $L_p$  of  $K^{(p)}$  and a map  $f_p: L_p^{(0)} \rightarrow \mathcal{A}$ ,  $v \mapsto A_v$  with

$$\overline{\text{St}}(v, L_p) \subset A_v$$

are given. We define a subdivision  $L_{p+1}$  of the  $(p+1)$ -skeleton of  $K$  and a map  $f_{p+1}$  as follows: For each  $(p+1)$ -simplex  $\sigma$  of  $K$ , let  $L_\sigma$  denote the subcomplex of  $L_p$  whose underlying space is  $\text{Bd } \sigma$ . Consider the cone  $\hat{\sigma} * L_\sigma$ . By Lemma 3.7, there is an  $N(\sigma) \in \mathbb{N}$  such that if we let

$$C(\sigma) = \text{sd}^{N(\sigma)}(\hat{\sigma} * L_\sigma, L_\sigma),$$

then the following holds: For each vertex  $v$  of  $C(\sigma)$  belonging to  $L_\sigma$ ,

$$\overline{\text{St}}(v, C(\sigma)) \subset A_v,$$

and for each vertex  $w$  of  $C(\sigma)$  not in  $L_\sigma$ , there exists an  $A \in \mathcal{A}$  such that

$$\overline{\text{St}}(w, C(\sigma)) \subset A.$$

Let  $L_{p+1}$  be the union of  $L_p$  and the complexes  $C(\sigma)$ , as  $\sigma$  ranges over all  $(p+1)$ -simplices of  $K$ .

If  $v \in L_p^{(0)}$ , then  $\overline{\text{St}}(v, L_{p+1})$  is the union of the sets  $\overline{\text{St}}(v, L_p)$  and  $\overline{\text{St}}(v, C(\sigma))$ , as  $\sigma$  ranges over the  $(p+1)$ -simplices of  $K$  containing  $v$ . By construction, each of these sets is a subset of  $A_v$ , so we define  $f_{p+1}(v)$  to be  $f_p(v)$ .

If  $w$  is a vertex of  $L_{p+1}$  not in  $L_p$ , then  $w$  lies in the interior of some  $(p+1)$ -simplex  $\sigma$  of  $K$ . Therefore

$$\overline{\text{St}}(w, L_{p+1}) = \overline{\text{St}}(w, C(\sigma)),$$

thus  $\overline{\text{St}}(w, L_{p+1})$  is contained in some  $A \in \mathcal{A}$ . We define  $f_{p+1}(w)$  to be such an element  $A_w$  of  $\mathcal{A}$ .

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The complex  $K'$  is defined to be the union of the complexes  $L_p$ . Let  $f: K'^{(0)} \rightarrow \mathcal{A}$  be defined as follows: If  $v \in K'^{(0)}$ , then  $v \in L_p^{(0)}$  for some  $p$ . Then let  $f(v) = f_p(v)$ . This is well-defined, for if  $v \in L_p^{(0)}$  and  $v \in L_q^{(0)}$ , then  $f_p(v) = f_q(v)$ , by definition.

If  $v \in K'^{(0)}$ , let  $p$  be the minimum number such that  $v \in L_p^{(0)}$ . Then

$$\overline{\text{St}}(v, L_{p+k}) \subset f_{p+k}(v) = f(v) = A_v$$

for all  $k \geq 0$ , and hence  $\overline{\text{St}}(v, K') \subset A_v$ . □

We are now finally able to prove the general simplicial approximation theorem.

*Proof of Theorem 3.2.* Let  $\mathcal{A}$  be the covering of  $|K|$  defined by

$$\mathcal{A} = \{h^{-1}(\text{St}(w, L)) \mid w \in L^{(0)}\}.$$

Choose a subdivision  $K'$  of  $K$  whose closed stars refine  $\mathcal{A}$ . Then  $h$  satisfies the star condition with respect to  $K'$  and  $L$ . □

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