**Block course on Spectral Sequences**

# Grothendieck Spectral Sequences

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**Definition 1** (Cartan-Eilenberg resolution)**.** Let $\mathcal{A}$ be an abelian category with enough projectives. A (left) *Cartan-Eilenberg resolution* $P_{\bullet,\bullet}$ of a chain complex $A_\bullet$ in $\mathcal{A}$ is an upper half-plane bicomplex consisting of projective objects, together with a chain map $\varepsilon\colon P_{\bullet,0} \to A_\bullet$ such that for every $p$

1. If $A_p = 0$, then $P_{p,\bullet} = 0$.
2. The maps on boundaries and homology

$$\begin{aligned} \mathrm{B}_p(\varepsilon)\colon\ \mathrm{B}_p(P, d^h) &\to \mathrm{B}_p(A) \\ \mathrm{H}_p(\varepsilon)\colon\ \mathrm{H}_p(P, d^h) &\to \mathrm{H}_p(A) \end{aligned}$$

   are projective resolutions in $\mathcal{A}$.

**Lemma 2.** *Let $\mathcal{A}$ be an abelian category with enough projectives. Every chain complex $A_\bullet$ has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \to A_\bullet$.*

*Proof.* For every $p$ we select projective resolutions $P^{\mathrm{B}}_{p,\bullet}$ of $\mathrm{B}_p(A)$ and $P^{\mathrm{H}}_{p,\bullet}$ of $\mathrm{H}_p(A)$. By the Horseshoe Lemma [Wei95, Lemma 2.2.8] there is a projective resolution $P^{\mathrm{Z}}_{p,\bullet}$ of $\mathrm{Z}_p(A)$ such that

$$0 \to P^{\mathrm{B}}_{p,\bullet} \to P^{\mathrm{Z}}_{p,\bullet} \to P^{\mathrm{H}}_{p,\bullet} \to 0$$

is a short exact sequence of chain complexes lying over

$$0 \to \mathrm{B}_p(A) \to \mathrm{Z}_p(A) \to \mathrm{H}_p(A) \to 0.$$

By applying the Horseshoe Lemma again, we find a projective resolution $P^{A}_{p,\bullet}$ of $A_p$ such that

$$0 \to P^{\mathrm{Z}}_{p,\bullet} \to P^{A}_{p,\bullet} \to P^{\mathrm{B}}_{p-1,\bullet} \to 0$$

is a short exact sequence of chain complexes lying over

$$0 \to \mathrm{Z}_p(A) \to A_p \to \mathrm{B}_{p-1}(A) \to 0.$$

Now we define $P_{\bullet,\bullet}$ to be the bicomplex with $P^{A}_{p,\bullet}$ as the $p$-th column. The vertical differential in the $p$-th column is the differential of $P^{A}_{p,\bullet}$ multiplied by $(-1)^p$. The horizontal differential is given by

$$P^{A}_{p+1,\bullet} \twoheadrightarrow P^{\mathrm{B}}_{p,\bullet} \hookrightarrow P^{\mathrm{Z}}_{p,\bullet} \hookrightarrow P^{A}_{p,\bullet}.$$

To see, that this construction gives a chainmap $\varepsilon$, consider the following diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P^{A}_{p,0} & \twoheadrightarrow & P^{\mathrm{B}}_{p-1,0} & \hookrightarrow & P^{\mathrm{Z}}_{p-1,0} & \hookrightarrow & P^{A}_{p-1,0} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varepsilon} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \\ \cdots & \longrightarrow & A_p & \twoheadrightarrow & \mathrm{B}_{p-1}(A) & \hookrightarrow & \mathrm{Z}_{p-1}(A) & \hookrightarrow & A_{p-1} & \longrightarrow & \cdots \end{array}$$

The inner square commutes because of the first application of the Horseshoe Lemma and the outer squares commute because of the second one.

From this diagram, we also see that $\mathrm{B}_p(P, d^h)$ is the chain complex $P^{\mathrm{B}}_{p,\bullet}$ with its differential multiplied by $(-1)^p$. Because $P^{\mathrm{B}}_{p,\bullet} \to \mathrm{B}_p(A)$ is a projective resolution, so is $\mathrm{B}(\varepsilon)$.

A similar diagram shows that $\mathrm{H}_p(P, d^h)$ is actually given by the chain complex $P^{\mathrm{H}}_{p,\bullet}$ with its differential multiplied by $(-1)^p$. Again, because $P^{\mathrm{H}}_{p,\bullet} \to \mathrm{H}_p(A)$ is a projective resolution, so is $\mathrm{H}(\varepsilon)$, which concludes the proof. □

**Remark 3.** By making the usual adjustments, we can define (right) *Cartan-Eilenberg resolutions* of cochain complexes in an abelian category $\mathcal{A}$. An equivalent of the above lemma holds (just use duality).

**Theorem 4** (Grothendieck Spectral Sequence)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Suppose we have left exact functors $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ such that $G$ sends injective objects of $\mathcal{A}$ to $F$-acyclic objects of $\mathcal{B}$. Then there exists a first quadrant cohomological spectral sequence for each $A$ in $\mathcal{A}$:*

$$E_2{}^{pq} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

*The edge maps in this spectral sequence are the natural maps*

$$(R^pF)(GA) \to R^p(FG)(A) \text{ and } R^q(FG)(A) \to F(R^qG(A)).$$

*The exact sequence of low degree terms is*

$$0 \to (R^1F)(GA) \to R^1(FG)A \to F(R^1G(A)) \to (R^2F)(GA) \to R^2(FG)A.$$

*Proof.* Choose an injective resolution $A \to I$ of $A$ in $\mathcal{A}$. By applying $G$, we get a cochain complex $G(I)$ in $\mathcal{B}$. Now let $G(I) \to J$ be a Cartan-Eilenberg resolution of $G(I)$. Now consider The bicomplex $F(J)$. Because G(I) is bouded below, this gives rise to two spectral sequences by using the usual filtrations on the total complex of $F(J)$. The first is given by

$${}^{I}E_2{}^{pq} = \mathrm{H}^p((R^qF)(GI)) \Rightarrow H^{p+q}(Tot(F(J))).$$

Because $GI^p$ is $F$-acyclic, $(R^qF)(GI) = 0$ for $q \neq 0$. Thus the spectral sequence collapses and yields

$$\mathrm{H}^p(Tot(F(J))) \cong \mathrm{H}^p(FG(I)) = R^p(FG)(A).$$

Therfore the second spectral sequence we get from the total complex of $J$ is given by

$${}^{II}E_2{}^{pq} = (R^pF)(R^qG(A)) = (R^pF)(\mathrm{H}^q(GI)) \Rightarrow \mathrm{H}^{p+q}(Tot(F(J))) \cong R^{p+q}(FG)(A),$$

which is the desired Grothendieck Spectral Sequence. □

**Remark 5.** An equivalent of the above Theorem holds for the homological case.

**Example 6** (Base-change for Tor)**.** Let $f\colon R \to S$ be a ring map and $B$ be an $S$-module. Now consider the composite

$$\mathrm{Mod}_R \xrightarrow{\bullet \otimes_R S} \mathrm{Mod}_S \xrightarrow{\bullet \otimes_S B} \mathrm{Ab}\,.$$

The functors $\bullet \otimes_R S$ and $\bullet \otimes_S B$ are right excat, because they are left adjoint functors. Additionally, because of the natural isomorphism $(A \otimes_R S) \otimes_S B \cong A \otimes_R B$, the $S$-module $A \otimes_R S$ is flat if and only if the $R$-module $A$ is flat. Projective $R$-modules are flat and flat $S$-modules are $(\bullet \otimes_S B)$-acyclic, so this implies that $\bullet \otimes_R S$ maps projective $R$-modules to $(\bullet \otimes_S B)$-acyclic $S$-modules. So the conditions of the above theorem are fullfilled and we get a first quadrant homological spectral sequence

$$E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), B) \Rightarrow L_{p+q}((\bullet \otimes_R S) \otimes_S B)(A) \cong \mathrm{Tor}^R_{p+q}(A, B).$$

**Example 7** (Base-change for Ext)**.** Let $f\colon R \to S$ be a ring map and $B$ be an $S$-module. Now consider the composite

$$\mathrm{Mod}_R \xrightarrow{\mathrm{Hom}_R(S,\bullet)} \mathrm{Mod}_S \xrightarrow{\mathrm{Hom}_S(B,\bullet)} \mathrm{Ab}\,.$$

The functors $\mathrm{Hom}_R(S, \bullet)$ and $\mathrm{Hom}_S(B, \bullet)$ are left exact. $\mathrm{Hom}_R(S, A)$ is an injective $S$-module if and only if the functor

$$\mathrm{Hom}_S(\bullet, \mathrm{Hom}_R(S, A)) \cong \mathrm{Hom}_R(\bullet \otimes_S S, A) \cong \mathrm{Hom}_R(\bullet, A)$$

is exact. This is the case if and only if $A$ an injective $R$-module. This shows that $\mathrm{Hom}_R(S, \bullet)$ maps injective $R$-modules to injective $S$-modules, i.e. to $\mathrm{Hom}_S(B, \bullet)$-acyclic $S$-modules. So the conditions of the above theorem are fullfilled and we get a first quadrant cohomological spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_S(B, \mathrm{Ext}^q_R(S, A)) \Rightarrow R^{p+q}(\mathrm{Hom}_S(B, \mathrm{Hom}(S, \bullet)))(A) \cong \mathrm{Ext}^{p+q}_R(B, A).$$

# Bibliography


[Wei95] C.A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995. ISBN: 9780521559874. URL: https://books.google.de/books?id=flm-dBXfZ%5C_gC.